Problem 1
One mole of a monoatomic ideal gas is compressed form an initial volume $V$ to a final volume $V/3$ at constant $T$. All answers given as a function of $T$ and $V$.

(a) What is the total energy $U_i$ of the gas before compression?

Use equipartition theorem. $f=1$.

$$U_i = \frac{f k_B T N}{2} = \frac{3k_B N_A T}{2}$$
• (b) What is the pressure $P_f$ of the compressed gas with $V=V/3$?

Use eq. of state, note $n=1$ mole.

\[ P_i V_i = NK_b T = nRT = P_f V_f \]

\[ P_f = \frac{nRT}{V_f} = \frac{3RT}{V} \]

• (c) How much work is done by the gas during compression?

\[ W = -\int_{V}^{V/3} P(V) dV = -\int_{V}^{V/3} \frac{RT}{V} dV = -RT \ln \frac{V/3}{V} \]

\[ = -RT \ln 1/3 = RT \ln 3 \]

The work is positive.
• (d) How much heat $Q$ is absorbed or released from the system during the compression?

Since $T_i = T_f$, we also have $U_i = U_f$ ($\Delta U = 0$), from the first Law: $\Delta U = Q + W = 0$, so $Q = -W$, from previous result $Q = -RT\ln 3$. This is heat that flows out of the system (to environment).

• (C) What is the entropy change $\Delta S$ that results from the compression?

We need to use Sackur-Tetrode eq. ($U =$ constant, $N =$ constant).

$$S(U, V, N) = Nk_B \ln \left( \frac{V}{N} \right) + \frac{3}{2} Nk_B \ln \left( \frac{4\pi mU}{3h^2 N} \right) + \frac{5Nk_B}{2}$$

$$\Delta S = NK_B \ln \frac{V_f}{V_i} = R \ln \frac{V/3}{V} = -R \ln 3$$
Two Einstein Solids. Solid A has \( N_A = 4 \) oscillators, solid B has \( N_B = 6 \) oscillators. They are placed in thermal contact and share a total of 5 units of energy \((q = 5)\).

(a) Construct a table with all the possible macropartitions of the combined system.

\[
\Omega(q, N) = \frac{q + (N - 1)!}{q!(N - 1)!} \\
\Omega_{AB} = \Omega_A \Omega_B \\
S/k = \ln \Omega
\]

<table>
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<th>( U_A )</th>
<th>( U_B )</th>
<th>( \Omega_A )</th>
<th>( S_A/k_B )</th>
<th>( \Omega_B )</th>
<th>( S_B/k_B )</th>
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</table>
• (b) Total number of microstates: 2002

• (c) Accurate plot of $S_A/k_B$ as a function of $U_A$ using table
• Estimate the temperature $T_A$ where solids A and B are in equilibrium.

At equilibrium $\Omega_{AB}$ is maximum ($U_A=2, U_B=3$).

$$\frac{1}{T_A} = \left. \frac{\Delta S}{\Delta U} \right|_{U=U_A} = \frac{S(3\epsilon) - S(1\epsilon)}{3\epsilon - 1\epsilon} \approx 0.8 \frac{k_B}{\epsilon}$$

$$T_A \approx 1.24\epsilon/k_B$$
Problem 3

In the box shown, isolated from the environment, the barrier is removed and the ideal gas expands adiabatically.

(a) work done by the gas during the expansion

\[ \Delta U = Q + W. \] But the system is isolated so there is no heat exchange with environment, \( Q = 0. \) Also, the \( \left< KE \right>/\text{molecule} \) is unaffected by the expansion, so \( \Delta U = 0. \) So this means that \( W = 0. \) This is an isothermal expansion, since \( U_i = U_f \) we also have \( T_i = T_f. \)

(b) What is the ratio of the pressures \( P_i / P_f \) before and after the expansion?

Using the ideal gas Eq. of State: \( P_i / P_f = V_f / V_i = 2 \)

Note that the eq \( P_i V_i^{\gamma} = P_f V_f^{\gamma} \) does not apply to this case cause this is an adiabatic isothermal process, but not isentropic process.
(c) What is the entropy change due to the expansion?

Use S-T eq. Note that N and U are constant.

\[ S(U, V, N) = Nk_B \ln \left( \frac{V}{N} \right) + \frac{3}{2} Nk_B \ln \left( \frac{4\pi mU}{3h^2 N} \right) + \frac{5Nk_B}{2} \]

\[ \Delta S = k_B \ln \frac{V_f}{V_i} = k_B \ln 2 \]

The entropy increases. This is because now the spatial degrees of freedom have increased (not the momentum hypersphere).
(d) A small leak opens and the gas slowly drains out. The container remains thermally isolated. What is the chemical potential associated with the escape of the gas?

Use the thermodynamic identity. Note that here V and U remain constant.

\[ dU = T dS|_{N,V} - P dV|_{U,N} + \mu dN|_{u,V} \]

\[ \Delta U = U_f - U_i = 0 - 3/2Nk_B T. \] However, note that U/N is constant (the KE per particle does not change, what changes is the number of particles).

For we can compute dS from S-T e.q. Note that in the thermodynamic identity it has to be computed for constant V and N. As also U/N is constant, \( \Delta S = 0 \).

\[ S(U, V, N) = Nk_B \ln \left( \frac{V}{N} \right) + \frac{3}{2} Nk_B \ln \left( \frac{4\pi mU}{3\hbar^2 N} \right) + \frac{5Nk_B}{2} \]

So dV|U,N=0, dS|N,V=0, which means that the chemical potential associated with this escape is:

\[ -\frac{3}{2} Nk_B T = \mu \Delta N = \mu (0 - N) \rightarrow \mu = 3/2k_N T \]