05/17/2013 PHY 306: Final Exam

Name

This exam is closed book. You are allowed 2 index cards not larger than 4"x6" (both sides writing allowed) with information of your own choosing. You are also allowed a calculator. The exam period is 11:15 AM-1:45 PM.

Some numerical constants:

\[ k_B = \text{Boltzmann constant} = 1.38 \times 10^{-23} \text{ J/K} \]
\[ K_B = 8.617 \times 10^{-5} \text{ eV/K} \]
\[ N_A = \text{Avogadro's number} = 6.02 \times 10^{23} \]

<table>
<thead>
<tr>
<th>Problem</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 1</td>
<td>35</td>
</tr>
<tr>
<td>Problem 2</td>
<td>30</td>
</tr>
<tr>
<td>Problem 3</td>
<td>30</td>
</tr>
<tr>
<td>Problem 4</td>
<td>30</td>
</tr>
<tr>
<td>Problem 5</td>
<td>35</td>
</tr>
<tr>
<td>Total</td>
<td>160</td>
</tr>
</tbody>
</table>
Problem 1 (35 points)

You have an Einstein solid with 3 oscillators and a two-state paramagnet with four spins. The magnetic field in the region of the paramagnet points "Down" and is carefully tuned so that $\mu B = 0.5\varepsilon$, where $\mu B$ is the energy of a spin pointing "up", $-\mu B$ is the energy of a spin pointing "down", and $2\varepsilon$ is the energy level separation of the oscillators. At the beginning of the experiment the energy in the Einstein solid $U_s$ is $6\varepsilon$ and the energy in the paramagnet $U_p$ is $-2\varepsilon$.

(a) (5 points) Using a schematic drawing of the Einstein solid, give an example of a microstate which corresponds to the macrostate $U_s = 6\varepsilon$.

(b) (5 points) Using a schematic drawing of the paramagnet, give an example of a microstate which corresponds to the macrostate $U_p = -2\varepsilon$.

(c) (10 points) Considering that the "system" comprises the solid and the paramagnet each of them with their corresponding initial energies ($U_s=6\varepsilon$ and $U_p=-2\varepsilon$) calculate the multiplicity of the system assuming that the solid and paramagnet cannot exchange energy.

$U_s = 6\varepsilon \Rightarrow 3$ particles
$U_p = -2\varepsilon \Rightarrow$ ground state $2p = 1$

$2s(3,3) = \frac{(3+3-1)!}{3!2!} = \frac{5!}{2!} = 10$
Now let the solid and paramagnet exchange energy until they come to thermal equilibrium.

(d) (5) What is the value of $U_s$ now? Draw an example of a microstate in which you might find the solid.

(e) (5) What is the value of $U_p$ now? Draw an example of a microstate in which you might find the paramagnet.

\[
\begin{array}{cccc}
N_s & N_p & U_p & U_s \\
6 \varepsilon & 10 & -2 \varepsilon & 10 \\
6 \varepsilon & 6 & 0 & 6 \varepsilon \\
2 \varepsilon & 3 & 2 \varepsilon & 1 \\
\end{array}
\]

This is the eq. state

\[
\Omega_p = \frac{N!}{N! \cdot N!}
\]

\[
\Omega \left( \frac{2, 3}{2+2} \right) = \frac{6}{2! \cdot 2!}
\]

(f) (5) What is the probability of finding the Einstein solid in a state with energy $U_s = 2 \varepsilon$? Why?

\[
P(U_s = 2 \varepsilon) = \frac{3}{36 + 3 + 10} = \frac{3}{49} \rightarrow \text{very low, because there are}
\]

\[
\text{probability of } \Omega_p = 1 \text{ is very small, given that energy can only be}
\]

\[
\text{exchanged in units of } \varepsilon \text{ spin excitations.}
\]
Problem 2 (30 points)

Two insulated tanks with volumes $V_1$ and $V_2$ are connected by a valve. $V_2$ is 10 times smaller than $V_1$. In $V_1$ there is 1 mole of a given monoatomic gas. At the initial state the gas in $V_1$ can be treated as an ideal gas. In $V_2$ there are 3 moles of the same monatomic gas, but is should be treated as a van der Waals gas in the initial state (the constants $a$ and $b$ are known, they are parameters in the problem). Initially, when the valve was closed, both gases were at the same initial temperature $T_i$.

The valve is open and the system reaches its equilibrium state. Note that the system is insulated, so no heat can be exchanged with the exterior.

(a). (10 pts) Find the final temperature $T_f$ once the system has reached equilibrium, assuming that in the final state the gas can be treated as an ideal gas.

\[
U_i = U_1 + U_2 = \frac{3}{2} \frac{NkT_i}{2} + \frac{33}{2} NkT_i - \left( \frac{3N^2a}{V_2} \right)
\]

\[
U_F = \frac{3}{2} \left( N + 3N \right) kT_F = 6NkT_F
\]

\[
U_i = U_F \Rightarrow T_F = \left( \frac{3NkT_i}{2} \left( 1 + 3 \right) - \frac{9N^2a}{V_2} \right) / 6Nk
\]

\[
T_F = T_i - \frac{9N^2a}{6kN} = T_i - \frac{3N^2a}{kN V_2}
\]
(b) (5 pts) Explain why your answer to part (a) depends on \( V_2 \) but not on \( V_1 \):

\[ \Rightarrow \text{because the gas in the smallest volume is a} \ \text{v} \ \text{mol} \ \text{gas (molecules interact with each other, reducing the total energy, i.e. the system is in a lower energy state).} \]

When the gas expands to all volume now it is an ideal gas, and \( V \) does not depend on \( V \).

(c) (10 pts) Calculate the change in entropy of the system from its initial state to its final state.

Initial state: \( S_i = S_1 + S_2 \)

\[ S_1 = 3Nk_B \ln \left( \frac{V_1}{N} \right) \left( \frac{27m}{h^2} \right) ^{3/2} \]

\[ S_2 = 3Nk_B \ln \left( \frac{V_1 + V_2}{N + 3N} \right) ^{3/2} \]

\[ S_{\text{final}} = S_{\text{ideal}} + Nk_B \ln \left( \frac{N}{V} \right) ^{3/2} \]

\[ S_F = 4 N k_B \ln \left( \frac{V_1 / V_2}{4N} \right) \left( \frac{27m}{h^2} \right) ^{3/2} \]

(d) (5 pts) Does it increase, decrease, or remain constant?

Why?

\[ \Delta S = S_F - S_i = \]

\[ = N k_B \ln \left[ \left( \frac{V_1}{N} \right)^{3/2} \left( \frac{V_1 + V_2}{N + 3N} \right)^{3/2} \right] + 3Nk_B \ln \left( \frac{V_1}{30N} \right)^{3/2} \]

\[ + 4 N k_B \ln \left( \frac{V_1 + V_2}{4N} \right) ^{3/2} \left( \frac{27m}{h^2} \right) ^{3/2} \]

\[ = \left\{ \begin{array}{l} 4 N k_B \ln \left( \frac{27m}{h^2} \right) ^{3/2} > 0 \hfill \\
4 N k_B \ln \left( \frac{27m}{h^2} \right) ^{3/2} < 0 \end{array} \right. \]
For Vol part: \( \Delta S \)

\[
- Nk_B \ln \left( \frac{V_1}{N} \right) - 3Nk_B \ln \left( \frac{V_i}{30N} \right) + Nk_B \ln \left( \frac{V_i + \frac{30}{4}}{4N} \right)
\]

\[
= Nk_B \ln \left( \frac{V_i}{4N} \right) - Nk_B \ln \left( \frac{V_i}{30N} \right) = Nk_B \ln \left( \frac{30}{4} \right) > 0
\]

For T part: \( \Delta S = 4Nk_B \ln \left( \frac{T_F}{T_i} \right)^{\frac{3}{2}} \) < 0

\( \frac{T_F}{T_i} = \frac{T_i - \frac{3N\alpha}{k_B}}{T_i} = 1 - \frac{3N\alpha}{k_B} \)

The result depends on the value of \( \alpha \), not physical. If \( \alpha < 0 \), then \( T_F < T_i \), but never remain constant. It is not physical, i.e., problem is not physical.
Problem 3 (30 points)

This problem is about partition functions.

The ground level of the neutral lithium atom is doubly degenerate (that is, \(d_0=2\)). The first excited level is 6-fold degenerate (\(d_1=6\)) and is at an energy 1.2 eV above the ground level. Consider the energy of the ground level to be zero.

Since all the other levels of Li are at a much higher energy, it is safe to neglect them for this problem, therefore this is a 2 level system, with degeneracy 2 for the lowest level and 6 for the highest level.

(a) (10) In the outer atmosphere of the Sun, which is at a temperature of about 6000K, what fraction of the neutral lithium is in the excited level?

\[
\begin{align*}
E &= 1.2 \text{ eV} \quad d_1 = 6 \\
1 - Z_{Li} &= \sum_i e^{-\beta E_i} = e^{-1.2/0.5} - e^{-1.2/1.2} \\
Z_{Li} &= 2 + 6 e^{-1.2/0.5} = 2 + 6 e^{-2.4} = 2.54 \\
P(E_1) &= \frac{6e^{-1.2/0.5}}{2 + 6 e^{-1.2/0.5}} = \frac{3}{e^{1.2/0.5} + 3} = 0.23
\end{align*}
\]

\(Z\) correct \(\rightarrow 5\) pts

(b) (5) Find the average energy of Li atom at temperature \(T\) (again, consider only the ground state and the first excited level).

\[
\langle E \rangle = \frac{1}{Z} \frac{\partial Z}{\partial \beta} = \frac{6e^{-1.2/0.5} e^{-\beta E_1}}{2 + 6 e^{-1.2/0.5}} = \frac{3 e^{-1.2/0.5}}{e^{1.2/0.5} + 3}
\]
(c) (15) Find the contribution of these levels to the specific heat $C_v$, and sketch $C_v$ as a function of $T$. What are the high $T$ and the low $T$ limits for $C_v$?

$$C_v = \left( \frac{\partial \langle E \rangle}{\partial T} \right)_V = -3 \epsilon \frac{\epsilon}{\epsilon + 3} \frac{\partial \epsilon}{\partial T} - k_B \frac{3 \epsilon^2}{\epsilon + 3} \frac{\epsilon}{(k_B T)^2}$$

**Problem 4 (30 points)**

The most probable speed is the maximum of the Maxwell Boltzmann distribution of speeds, call it $V_{\text{max}}$. The root-mean-squared speed ($V_{\text{rms}}$) is the speed for which the kinetic energy of a particle is equal to $3/2k_B T$.

(a) (10) Derive $V_{\text{max}}$ and $V_{\text{rms}}$.

$$V_{\text{rms}} \rightarrow \epsilon_n = \frac{1}{z} m V_{\text{rms}}^2 = \frac{3}{2} k_B T \Rightarrow V_{\text{rms}} = \sqrt{\frac{3 k_B T}{m}}$$

$$V_{\text{max}} \rightarrow \frac{\partial P(V)}{\partial V} = 0 \Rightarrow P(V) = \left( \frac{2}{\pi} \right)^{1/2} \frac{1}{(m k_B T)^{3/2}} V^2 e^{-\frac{m k_B V^2}{2T}}$$

$$\frac{\partial P(V)}{\partial V} = e^{-\frac{m k_B V^2}{2T}} \left( -\frac{m k_B V^2}{2T} + V^2 \right) = 0 \Rightarrow V = \frac{2 k_B T}{m}$$
(b) (10) Find the temperature $T$ at which $V_{\text{rms}}$ of a hydrogen molecule $\text{H}_2$ exceeds $V_{\text{max}}$ by 400 m/s.

\[
V_{\text{rms}} = \sqrt{\frac{3kT}{m}} \quad V_{\text{max}} = \sqrt{\frac{2kT}{m}}
\]

\[
V_{\text{rms}} - V_{\text{max}} = 400 \text{ m/s}
\]

\[
T \frac{1}{2} (\sqrt{\frac{3kT}{m}} - \sqrt{\frac{2kT}{m}}) = 400 \text{ m/s} \rightarrow T = \left(\frac{400 \text{ m/s}}{(\sqrt{\frac{3kT}{m}} - \sqrt{\frac{2kT}{m}})}\right) = \left(\frac{400}{\text{111.33-98.90}}\right) = 383 \text{ K}
\]

(c) (10) The earth's escape velocity (the velocity an object must have at the sea level to escape the earth's gravitational field) is $7.9 \times 10^3$ m/s. Compare this velocity with the root mean square thermal velocity at 300K of (a) a nitrogen molecule $\text{N}_2$ and (b) a hydrogen molecule $\text{H}_2$. Explain why the earth's atmosphere contains nitrogen but not hydrogen.

\[
M(\text{H}) = 1 \text{ g/mol} \\
M(\text{N}) = 14 \text{ g/mol}
\]

\[
V_{\text{rms}}(\text{N}_2) = \sqrt{\frac{3kT}{28 \times 1.67 \times 10^{27}}} = 461 \text{ m/s}
\]

\[
V_{\text{rms}}(\text{H}_2) = \sqrt{\frac{3kT}{2 \times 1.67 \times 10^{27}}} = 1928 \text{ m/s}
\]

\[
\frac{V_{\text{rms}}(\text{N}_2)}{V_{\text{escape}}} = 6.5\% \quad \text{Vescape} \\
\frac{V_{\text{rms}}(\text{H}_2)}{V_{\text{escape}}} = 2.4\% \quad \text{Vescape}
\]

There is still some gas in the tail of the distribution that escapes the atmosphere.
Problem 5 (35 points)

Constants that you will need:

\[ m_e = 9.1 \times 10^{-31} \text{ Kg (mass of electrons)} \]
\[ m(\text{Li}) = 11.5 \times 10^{-27} \text{ kg (mass of Li atoms)} \]
\[ K_b = 8.617 \times 10^{-5} \text{ eV/K} \]
\[ 1\text{eV} = 1.6 \times 10^{-19} \text{ J} \]

The density of Magnesium atoms is \( 4.36 \times 10^{28} \text{ m}^{-3} \)

Each Mg atom has 2 conducting electrons (two electrons per atom contribute to the electronic fermi gas).

(a) (5) compute the fermi energy of Mg

\[
E_F = \frac{\hbar^2}{8m} \left( \frac{3}{\pi} \right)^{2/3} \quad n_e = 4.36 \times 10^{28} \text{ m}^{-3}
\]

\[
2 \times 8.72 \times 10^{-18} \text{ J} = 7.18 \text{ eV}
\]

(b) (10) The electrons participate in the current flow if their energies correspond to the occupancy \( n(\varepsilon) \) that is not too close to 1 (no empty states available for the accelerated electrons) and not too small (no electrons to accelerate). Assume that for these electrons the occupancy varies between 0.1 and 0.9. Calculate the energy interval that is occupied by the electrons that participate in the current flow at \( T = 50 \text{ K} \) and at \( T = 400 \text{ K} \).

\[
\Delta(50 \text{ K}) = 2k_b \ln \frac{9}{1} = 0.019 \text{ eV}
\]
\[
\Delta(400 \text{ K}) = 2k_b \ln \frac{9}{1} = 0.151 \text{ eV}
\]
(c)(10) Using the assumptions in (b) calculate the ratio $N(50K)/N(400K)$ where $N$ is the of the number of "current carrying electrons". To obtain $N$ you need assume that the density of states in the region where $n(\varepsilon)$ changes between 0.1 and 0.9 is constant and equal to the density of states at the Fermi level. Also that between these two values $n(\varepsilon)$ varies linearly. The density of states of the three dimensional Fermi Gas is:

$$g(\varepsilon) = \frac{3N}{2E_F^{3/2}} \sqrt{\varepsilon}$$

$$N_i(T=50K) = \int_{E_F - \Delta E/2}^{E_F + \Delta E/2} n(\varepsilon) g(\varepsilon) d\varepsilon = \frac{3N}{2E_F^{3/2}} \sqrt{E_F} \cdot \frac{1}{2} \Delta E(50K)$$

$$N_i(T=400K) = \frac{3N}{2E_F^{3/2}} \sqrt{E_F} \cdot \frac{1}{2} \Delta E(400K)$$

$$\frac{N(50K)}{N(400K)} = \frac{0.019}{0.151} = 0.125$$

(c)(10) Lets assume that degenerate Fermi gas becomes a classical boltzman gas when $T/E_F > 0.5$. Calculate this classical/quantum transition temperature for the fermi gas of electrons Mg and for a gas of Li atoms (Li atoms are fermions) of density $10^{24} m^{-3}$. Explain why these two temperatures are so different.

$$T > 0.5 \Rightarrow \frac{E_F}{k_B} \Rightarrow T = \frac{E_F}{k_B}$$

$$E_F for Mg = \frac{\hbar^2}{8m} \left( \frac{3}{N} - \frac{1}{N} \right) = \frac{6.6 \times 10^{-34} \ s^2}{11.5 \times 10^{-27} m \cdot 8} = 4.95 \times 10^{-26} eV$$

$$E_F for Li atoms = \frac{\hbar^2}{8m} \left( \frac{3}{N} \right) = \frac{6.6 \times 10^{-34} \ s^2}{11.5 \times 10^{-27} m \cdot 8} = 2.87 \times 10^{-7} eV$$

$$T_{Mg} > 0.5 \Rightarrow 2.87 \times 10^{-7} eV / 8.617 \times 10^{-5} eV_K = 1.6 \times 10^{-3} K$$

$$T_{Li} > 0.5 \Rightarrow 2.87 \times 10^{-7} eV / 8.617 \times 10^{-5} eV_K = 1.6 \times 10^{-3} K$$

They are so different because the degress primarily are in $e^-$ pair of cold $Joule$ (very low density) and those other are in a gas of $e^-$ high density. The answer is also important.