Uncertainty in measurements

In Physics, like every other experimental science, one cannot make any measurement without having some degree of uncertainty. In reporting the results of an experiment, it is as essential to give the uncertainty, as it is to give the best-measured value. Thus it is necessary to learn the techniques for estimating this uncertainty. Although there are powerful formal tools for this, simple methods will suffice for us. To a large extent, we emphasize a “common sense” approach based on asking ourselves just how much any measured quantity in our experiments could be in error.

A frequent misconception is that the experimental error is the difference between our measurement and the accepted “official” value. What we mean by error is the estimate of the range of values within which the true value of a quantity is likely to lie. This range is determined from what we know about our lab instruments and methods. It is conventional to choose the error range as that which would comprise 68% of the results if we were to repeat the measurement a very large number of times.

In fact, we seldom make enough repeated measurements to calculate the error precisely, so the error is usually an estimate of this range. Note, however, that the error range is established so as to include most of the likely outcomes, but not all of them. You might think of the process as a wager: pick the range so that if you bet on the outcome being within your error range, you will be right about 2/3 of the time. If you underestimate the error, you will lose money in your betting; if you overestimate it, no one will take your bet!

Error

If we denote a quantity that is determined in an experiment as $X$, we can call the error $\Delta X$. Thus if $X$ represents the length of a book measured with a meter stick we might say the length $l = 25.1 \pm 0.1$ cm where the central value for the length is 25.1 cm and the error, $\Delta l$ is 0.1 cm. Both the central value and error of measurements must be quoted when reporting your results. Note that in this example, the central value is given with just three significant figures. Do not write significant figures beyond the first digit of the error on the quantity. Giving more precision to a value than this is misleading and irrelevant.

Absolute Error

An error such as that quoted above for the book length is called the absolute error; it has the same units as the quantity itself (cm in the example). Note that if the quantity $X$ is multiplied by a constant factor $a$ the absolute error of $(aX)$ is:

$$\Delta(aX) = a\Delta X$$

(1.1)
Relative Error

We will also encounter **relative error**, defined as the ratio of the error to the central value of the quantity so that the

\[
\text{relative error of } X = \frac{\Delta X}{X}
\]  

(1.2)

Thus the relative error of the book length is \(\Delta l/l = (0.1/25,1) = 0.004\). The relative error is dimensionless, and should be quoted with as many significant figures as are known for the absolute error. Note that if the quantity \(X\) is multiplied by a constant factor \(a\) the relative error of \((aX)\) is the same as the relative error of \(X\),

\[
\frac{\Delta(aX)}{aX} = \frac{\Delta X}{X}
\]  

(1.3)

since the constant factor \(a\) cancels in the relative error of \((aX)\). Note that quantities with assumed negligible errors are treated as constants.

You are probably used to the **percentage error** from everyday life. The percentage error is the relative error multiplied by 100.

**Changing from a relative to absolute error:**

Often in your experiments you have to change from a relative to an absolute error by multiplying the relative error with the central value,

\[
\Delta X = \frac{\Delta X}{X} \times X
\]  

(1.4)

Random Error

Random error occurs because of small random variations in the measurement process. For example, measuring the time of a pendulum's period with a stopwatch will give different results in repeated trials due to small differences in your reaction time in hitting the stop button as the pendulum reaches the end point of its swing. If this error is random, the average period over the individual measurements would get closer to the correct value as the number of trials \(N\) is increased. The correct reported result would be the average for our central value,

\[
\overline{t} = \frac{\sum t_i}{N}
\]  

(1.5)
The error is usually taken as the standard deviation of the measurements. (In practice, we seldom take the trouble to make a very large number of measurements of a quantity in this lab.) An estimate of the random error of a single measurement \( t_i \) is

\[
\Delta t = \sqrt{\frac{\sum (t_i - \bar{t})^2}{N}}
\]  
\[ (1.5a) \]

and of the error of the average \( \bar{t} \) is

\[
\Delta \bar{t} = \sqrt{\frac{\sum (t_i - \bar{t})^2}{N(N-1)}}
\]  
\[ (1.5b) \]

where the sum is over the \( N \) measurements \( t_i \). Note in equation (1.5b) the “bar” over the letter \( t \), indicating that the error refers to the average \( t \).

In the case that we only have one measurement, but know, (from a previous measurement), what the error of the average is, we can use this error of the average \( \bar{t}, \Delta \bar{t} \), multiplied by \( \sqrt{N - 1} \) as the error of this single measurement (which you see when you divide equation (1.5a) by equation (1.5b).)

If we don’t have a value of the error of \( \bar{t}, \Delta \bar{t} \), we must guess the likely variation from the character of your measuring equipment. For example in the book length measurement with a meter stick marked off in millimeters, you might guess that the error would be about the size of the smallest division on the meter stick (0.1 cm).

**Systematic Error**

Some sources of uncertainty are not random. For example, if the meter stick that you used to measure the book was warped or stretched, you would never get a good value with that instrument. More subtly, the length of your meter stick might vary with temperature and thus be good at the temperature for which it was calibrated, but not others. When using electronic instruments such as voltmeters and ammeters, you obviously rely on the proper calibration of these devices. But if the student before you dropped the meter, there could well be a systematic error. Estimating possible errors due to such systematic effects really depends on your understanding of your apparatus and the skill you have developed for thinking about possible problems. For example if you suspect a meter might be mis-calibrated, you could compare your instrument with a ‘standard’ meter – but of course you have to think of this possibility yourself and take the trouble to do the comparison. In this course, you should at least consider such systematic effects, but for the most part you will simply make the assumption that the systematic errors are small. However, if you get a value for some quantity that seems rather far off what you expect, you should think about such possible sources more carefully.
Propagation of Errors

Often in the lab, you need to combine two or more measured quantities, each of which has an error, to get a derived quantity. For example, if you wanted to know the perimeter of a rectangular field and measured the length $l$ and width $w$ with a tape measure, you would then have to calculate the perimeter, $p = 2(l + w)$, and would need to get the error of $p$ from the errors you estimated for $l$ and $w$, $\Delta L$ and $\Delta W$. Similarly, if you wanted to calculate the area of the field, $A = lw$, you would need to know how to do this using $\Delta L$ and $\Delta W$.

There are simple rules for calculating errors of such combined, or derived, quantities. Suppose that you have made primary measurements of quantities $A$ and $B$, and want to get the best value and error for some derived quantity $S$.

For addition or subtraction of measured quantities the absolute error of the sum or difference is the 'addition in quadrature' of the absolute errors of the measured quantities, if $S = A \pm B$,

$$\Delta S = \sqrt{(\Delta A)^2 + (\Delta B)^2}$$

(1.6)

This rule, rather than the simple linear addition of the individual absolute errors, incorporates the fact that random errors (equally likely to be positive or negative) partly cancel each other in the error $\Delta S$.

For multiplication or division of measured quantities the relative error of the product or quotient is the 'addition in quadrature' of the relative errors of the measured quantities, if $S = A \times B$ or $\frac{A}{B}$,

$$\frac{\Delta S}{S} = \sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2}$$

(1.7)

Due to the quadratic addition in (1.6) and (1.7) one can often neglect the smaller of two errors. For example, if the error of $A$ is 2 (in arbitrary units) and the error of $B$ is 1, then the error of $S = A + B$ is

$$\Delta S = \sqrt{(\Delta A)^2 + (\Delta B)^2} = \sqrt{2^2 + 1^2} = \sqrt{5} = 2.23.$$ 

Thus, if you don’t want to be more precise in your error estimate than $\sim 12\%$ (which in most cases is sufficient, since errors are an estimate and not a precise calculation) you can simply neglect the error in $B$, although it is 1/2 of the error of $A$.

For the power $A^n$ of the measured quantity $A$ the relative error of the power is the relative error of $A$ multiplied by the magnitude of the exponent $n$, if $S = A^n$,

$$\frac{\Delta S}{S} = |n| \times \frac{\Delta A}{A}$$

(1.8)
An experiment with the simple pendulum

Often you will be asked to graph results obtained in the lab and to find certain quantities from the slope of the graph. You will always plot the quantities against one another in such a way that you end up with a linear plot. It is important to have error bars on the graph which reflect the uncertainty in the quantities you are plotting and help you to estimate the error in the slope of the graph, and hence the error in the quantity you are trying to find.

To demonstrate this we are going to consider an example that you will study in detail later in the course, the simple pendulum. A simple pendulum consists of a weight $w$ suspended from a fixed point by a string of length $L$. The weight swings about a fixed point. At a given time, $\theta$ is the angle which this string makes relative to the vertical (direction of the force of gravity).

![Diagram of a simple pendulum](image)

The period of this motion is defined as the time $T$ necessary for the weight to swing back and forth once. You will learn later in Chapter 9 on oscillations that an approximate relation between the period $T$ and length $L$ of the pendulum is given by $\displaystyle T = 2\pi \sqrt{\frac{L}{g}}$ where $g$ is the constant acceleration of gravity $g = 9.81 \text{ms}^{-2}$. In the derivation of this equation in Chapter 9 the assumption is made that the angle $\theta$ is small.

By measuring the period of oscillation of the pendulum as a function of the length in the string we can find a value for the acceleration due to gravity. The video shows you how we measure the different quantities that are important in the experiment, the length of the string, $L$, the angle, $\theta$ and the period of oscillation, $T$. Note that we measure the length several times and the period for 10 oscillations to try to minimize random errors. Notice that we use the computer as a stopwatch: we will be using the computer frequently in this course to make measurements and record data.

**Video**
Equipment

- Pendulum: 1 steel ball, 1 holder, 1 string
- Protractor (to measure angles)
- Computer: to be used as a simple timer
- Ruler: (to measure length)

Estimating the main errors in the experiment

Error in the length of the string

Suppose that we measure the length of the string 5 times and find the following 5 values for the length of the string, L.

<table>
<thead>
<tr>
<th>Measurement No.</th>
<th>L [cm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>56.4</td>
</tr>
<tr>
<td>2</td>
<td>56.6</td>
</tr>
<tr>
<td>3</td>
<td>56.7</td>
</tr>
<tr>
<td>4</td>
<td>56.6</td>
</tr>
<tr>
<td>5</td>
<td>56.5</td>
</tr>
</tbody>
</table>
Finding the average value is straightforward:

$$\bar{L} = \frac{56.4+56.6+56.7+56.6+56.5}{5} = 56.56 \text{ cm}$$

You find the error in the average using equation (1.5b):

$$\Delta L = \sqrt{\frac{(56.4-56.56)^2+(56.6-56.56)^2+(56.7-56.56)^2+(56.6-56.56)^2+(56.5-56.56)^2}{5 \times (4)}} = 0.05 \text{ cm}$$

So we can say that our measured value for $L$ is $56.56 \pm 0.05$ cm.

You should now do this yourself (for a different length of the string!). Choose a length at random. Measure it 5 times and enter the values on your report sheet. Now find the average value and the error in the average value.

**Error in the period**

If we measure the time for 10 oscillations we can find the time for one oscillation simply by dividing by 10. Now we need to make an estimate of the error.

First you need to estimate the error in your measurement. How accurately do you think you can press the button to tell the computer when to start and stop the measurement? Let's say that you think you can press the button within 0.2 seconds of either the start or the stop of the measurement. You need to account for the errors both times, but as we discussed earlier, because these errors are random they add in quadrature so you can say that

$$\Delta (10T) = \sqrt{0.2^2 + 0.2^2} = 0.28 \text{ s}$$

Now we find the error in $T$ by dividing by 10

$$\Delta T = \frac{\Delta (10T)}{10} = \frac{0.28}{10} = 0.028 \text{ s}$$

So you can see it was a good idea to measure several periods instead of one, we get a much more accurate result. Maybe you'd like to think about why we don't measure 100 oscillations (and because you'd get bored is only part of the answer!).

How accurately do you think you can press the button, is an 0.2 seconds and overestimate or underestimate for your reaction time? If you think that the accuracy of your button press is different to 0.2 seconds you should work out what you think $\Delta T$ is if you make a measurement of 10 oscillations.

**The effect of angle on the period of oscillation**

We mentioned above that the equation we want to test is only valid for small angles. The first measurement we will make is to measure the period of oscillation for 3 angles of release.

Choose three angles at $\sim 15^\circ$, $\sim 30^\circ$ and $\sim 80^\circ$ and measure the time for 10 periods for each angle. Enter your data into the table on your worksheet. You measure the time using the computer as a simple timer: Double click the Timer Program Icon. The Timer Program comes up. It has a “Start” “Stop” toggle button.

Taking into account your estimate for the error in $T$ above are the periods the same for all 3 angles. Note that experimentally two measurements are considered to be the same if their is an overlap of their range taking into account the error of their measurement. e.g $15^\circ \pm 1^\circ$ is equal within error to $16^\circ \pm 1^\circ$, $13^\circ \pm 1^\circ$ is not.
Taking data

Now we have some idea about the errors involved in our experiment we should take some data. You will need to measure the period of oscillation of the pendulum for different values of the string length L. You will want to have the same angle $\theta$ for each L. So the procedure you should follow is to set your string length to a given L, which you should write in the table on your worksheet, and then pull it back to angle of 15° which you can measure with the protractor. After releasing the pendulum you should measure, using the computer stopwatch, the time for 10 oscillations and record that on the worksheet as well. Do this for 10 different lengths of the string L. Once you have your data for 10T you can then work out what $T$ is for each L simply by dividing by 10. You should also enter in to your table values for $\Delta L$ and $\Delta T$ based on the estimates you made earlier.

Making a plot of our data

Recall that we said earlier that we expect that $T = 2\pi \sqrt{\frac{L}{g}}$. We can rearrange this as $L = \frac{g}{(2\pi)^2} T^2$, which means that we should get a straight line if we plot $L$ against $T^2$. To test this we need to work out what $T^2$ is for each value of $L$ and of course we need to know what the error in $T^2$, $\Delta T^2$, is so that we can draw error bars on the graph. So you need to complete the last to columns of your table. Finding $T^2$ is easy enough. To obtain $\Delta T^2$ we will need to propagate the error in $T$, $\Delta T$, using some of the equations above. Equation (1.8) tells us how to propagate the error of a quantity in to the error of the power of that quantity. In this case the relative error in $T^2$, which is $\frac{\Delta T^2}{T^2}$, is the same as twice the relative error in $T$, which is $\frac{\Delta T}{T}$. You can thus find that $\Delta T^2 = 2T\Delta(T)$. Make sure you can understand how to get this equation, your TA may ask you to show how it is obtained!

Once you have completed your table we can use the plotting tool. It’s built right in to the webpage, but when you enter your data and click “submit” it will make the graph in a new tab. This makes it easy to change something and get another graph if you made a mistake. You should enter the $T^2$ values as your x values and your $L$ values as your y values. According to the equation we are testing when $T^2 = 0$, $L = 0$ so you should check the box which asks you if the fit goes through (0,0). Enter the appropriate errors in the +/- boxes and choose “errors in x and y”. Click “submit” when you are done.

x axis label (include units):

y axis label (include units):

Check this box if the fit should go through (0,0).

(Don't include (0,0) in your list of points below, it will mess up the fit.)

What kind of errors are you entering below? None

<table>
<thead>
<tr>
<th>x1: +/-</th>
<th>y1: +/-</th>
</tr>
</thead>
<tbody>
<tr>
<td>x2: +/-</td>
<td>y2: +/-</td>
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<tr>
<td>x3: +/-</td>
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<td>x5: +/-</td>
<td>y5: +/-</td>
</tr>
<tr>
<td>x6: +/-</td>
<td>y6: +/-</td>
</tr>
</tbody>
</table>
If you entered everything right then on your new tab you should see a plot that looks something like this:

The data is clearly quite linear when plotted like this, so it gives us an indication that our formula at the least has the right form. (Maybe you would like to try plotting $L$ directly against $T$ and see what looks like). Notice that you can’t see the y error bars because they are very small. The program has fitted the data using a least-squares fitting approach. This means that it has calculated for each data point the square of the difference between the data point and the line. It then adds up all these “squares” and uses this number to determine how good the fit is. The computer tries to find the line that gives the smallest sum of squares and calls this the line of best fit. It's drawn this on the graph and called it “$y = a \times x$”. It's also given you the value of $a$ and its estimate for the uncertainty in $a$. The value the program gives for the error in $a$ is often fairly small, it relies mostly on the scatter in $a$ and only uses the errors you enter to weight the points differently in its fit.

Another technique you can use to estimate the error in the slope is to draw “max–min lines”. Here we draw in two lines, one that has the maximum slope that seems reasonable, the “max” line, and another with the smallest slope that seems reasonable, the “min” line. Normally we do these exercises on paper, but you can probably do it simply by holding a clear plastic ruler up to the screen to decide where you think the max–min lines should be (please DON'T draw on the screen!!). A line is reasonable if it just passes within most of the error bars. You then just take two convenient points on the line, and find the change in $y$ over the change in $x$ to calculate the slope. You can then work out the slope of both lines to give yourself an estimate of the error.
in the slope. In the example below you can calculate that the “max” line has a slope of about $90/3.6 = 25 \text{ cm/s}^2$, and the “min” line has slope of about $90/3.8 = 23.7 \text{ cm/s}^2$, therefore if you used this method you would conclude that the value of the slope is $24.4 +/ -0.7 \text{ cm/s}^2$, as compared to the computers estimate of $24.41 +/ -0.16 \text{ cm/s}^2$. Note that I drew the lines through (0,0) which we can consider as an error free point, i.e. the fits must go through this point.

Now we have a value from the slope we can calculate a value for the acceleration due to gravity, $g$, from it. Remember that $L = \frac{g}{(2\pi)^2}T^2$. This means that the slope of our graph should be equal to $\frac{g}{(2\pi)^2}$. To get $g$ we should multiply the slope by $(2\pi)^2$ and we should also divide by 100 to convert from cm/s$^2$ to m/s$^2$ which are the standard SI units. Using the computers values we find:

$$g = 9.64 +/ -0.06 \text{ m/s}^2$$

or using the max/min line estimate of the error we find:

$$g = 9.64 +/ -0.28 \text{ m/s}^2$$

The accepted value for $g$ is $9.81 \text{ m/s}^2$, which falls within the range we found using the max/min method and so we can say, based on that estimate, that our example experiment was consistent with the equation we tried to fit the data with. If we used the computers error estimate then we would conclude that the data was inconsistent with the textbook value. This demonstrates why we need to be careful about the methods we use to estimate error, depending on the data one method may be better than the other. Generally it is safer to take the larger of the two error estimates, but these kinds of judgments are the kinds of things it will be useful to discuss with your TA when you are doing experiments and analyzing your data. Another thing to bear in mind is that we were quite careful here about trying to eliminate random error, but if systematic error
was present then our methods would not have done very much to help. This is always something we should bear in mind when comparing values we measure in the lab to textbook values. We also need to think about simplifying assumptions we make. For example, we assumed that the pendulum did not slow down at all over the 10 swings we measured, and there are also approximations in the derivation of the equation we test here. Bearing these things in mind, an important point to make is that in general we should not necessarily be surprised if something we measure in the lab does not match exactly with what we might expect. When things don't seem to work we should think about why they don't, but, most importantly of all, we must never modify our data to make it match our expectations!

You should now work out the value of $g$ from the plot of your own data. If it is not consistent with the accepted value you should think about why this might be. Discuss these issues with your TA!

Once you have your value of $g$ we should also make a plot of $L$ directly against $T$.

x axis label (include units):

y axis label (include units):

Check this box if the fit should go through $(0,0)$. ☐

(Don't include $(0,0)$ in your list of points below, it will mess up the fit.)

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</table>

Was this plot linear? Do you think that based on the difference between the graphs you are able to confirm what the correct equation is that relate the period of oscillation to the length of the string?

Finishing up

Once you have finished your experiment you need to discuss your results with your TA. Don't leave until they say you can as you only get credit for the lab if the TA thinks you did a good enough job on the experiment and have mastered the key points that the experiment covers!